OCR
Oxford Cambridge and RSA

Tuesday 18 June 2019 – Morning

A Level Further Mathematics A

Y543/01 Mechanics

Time allowed: 1 hour 30 minutes

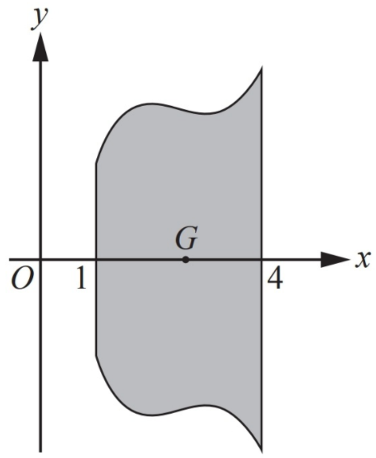
You must have:
- Printed Answer Booklet
- Formulae A Level Further Mathematics A

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

1 The region bounded by the x-axis, the curve $y = \sqrt{2x^3 - 15x^2 + 36x - 20}$ and the lines $x = 1$ and $x = 4$ is rotated through 2π radians about the x-axis to form a uniform solid of revolution R. The centre of mass of R is the point G (see diagram).



(a) Explain why the y-coordinate of G is 0. **[1]**

(b) Find the x-coordinate of G. **[4]**

P is a point on the edge of the curved surface of R where $x = 4$. R is freely suspended from P and hangs in equilibrium.

(c) Find the angle between the axis of symmetry of R and the vertical. **[3]**

a. R is symmetrical about the x-axis, and G lies on the line of symmetry of R

b. x-coord $= \dfrac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$

denominator: $\int_1^4 \left(\sqrt{2x^3 - 15x^2 + 36x - 20}\right)^2 dx$

$= \left[\frac{x^4}{2} - 5x^3 + 18x^2 - 20x\right]_1^4$

$= \left(\frac{4^4}{2} - (5 \times 4^3) + (18 \times 4^2) - (20 \times 4)\right) - \left(\frac{1^4}{2} - (5 \times 1^3) + (18 \times 1^2) - (20 \times 1)\right)$

$= 16 - \frac{-13}{2}$

$= \frac{45}{2}$

numerator : $\int_1^4 x\left(\sqrt{2x^3-15x^2+36x-20}\right)^2 dx$

$$= \int_1^4 \left(2x^4-15x^3+36x^2-20x\right) dx$$

$$= \left[\frac{2}{5}x^5 - \frac{15}{4}x^4 + 12x^3 - 10x^2\right]_1^4$$

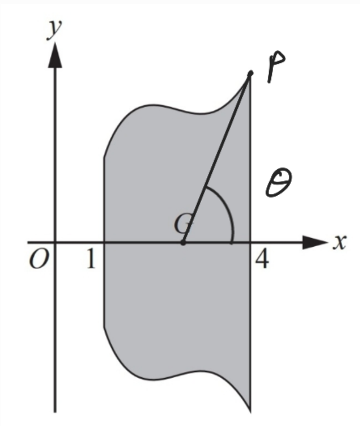
$$= \frac{2}{5}\times4^5 - \frac{15}{4}\times4^4 + 12\times4^3 - 10\times4^2 - \left(\frac{2}{5} - \frac{15}{4} + 12 - 10\right)$$

$$= \frac{288}{5} - -\frac{27}{20}$$

$$= \frac{1179}{20}$$

$$\therefore \bar{x} = \frac{1179 \div 20}{45 \div 2} = \frac{131}{50}$$

c.



at P : $x = 4$

$$y = \sqrt{(2\times4^3) - (15\times4^2) + (36\times4) - 20}$$

$$= 2\sqrt{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2\sqrt{3}}{4-2.62}\right)$$

$$= 1.19 \text{ rad}$$

2 A solenoid is a device formed by winding a wire tightly around a hollow cylinder so that the wire forms (approximately) circular loops along the cylinder (see diagram).

When the wire carries an electrical current a magnetic field is created inside the solenoid which can cause a particle which is moving inside the solenoid to accelerate.

A student is carrying out experiments on particles moving inside solenoids. His professor suggests that, for a particle of mass m moving with speed v inside a solenoid of length h, the acceleration a of the particle can be modelled by a relationship of the form $a = km^{\alpha}v^{\beta}h^{\gamma}$, where k is a constant. The professor tells the student that $[k] = \mathrm{MLT}^{-1}$.

(a) Use dimensional analysis to find α, β and γ. **[6]**

(b) The mass of an electron is 9.11×10^{-31} kg and the mass of a proton is 1.67×10^{-27} kg.

For an electron and a proton moving inside the same solenoid with the same speed, use the model to find the ratio of the acceleration of the electron to the acceleration of the proton. **[3]**

(c) The professor tells the student that a also depends on the number of turns or loops of wire, N, that the solenoid has.

Explain why dimensional analysis **cannot** be used to determine the dependence of a on N. **[1]**

a. $[a] = LT^{-2}$

$[h] = L$

$[m] = M$

$[v] = LT^{-1}$

Subbing into $a = km^{\alpha}v^{\beta}h^{\gamma}$:

$LT^{-2} = MLT^{-1}M^{\alpha}L^{\beta}T^{-\beta}L^{\gamma}$

Powers on LHS must = powers on RHS:

M: $0 = 1 + \alpha \Rightarrow \underline{\alpha = -1}$

$T: \quad -2 = -1 - \beta \quad \Rightarrow \underline{\beta = 1}$

$L: \quad 1 = 1 + \beta + \gamma$

$\quad\;\; 1 = 1 + 1 + \gamma \quad \Rightarrow \underline{\gamma = -1}$

b. electron: $a = k \times (9.11 \times 10^{-31})^{-1} \times v \times h$

$= 1.10 \times 10^{30}\, kvh$

proton: $a = k \times (1.67 \times 10^{-27})^{-1} \times v \times h$

$= 5.99 \times 10^{26}\, kvh$

ratio: $1.10 \times 10^{30} : 5.99 \times 10^{26}$

$1836 : 1$

c. N is dimensionless

3 A particle Q of mass m kg is acted on by a single force so that it moves with constant acceleration $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mathrm{m\,s^{-2}}$. Initially Q is at the point O and is moving with velocity $\mathbf{u} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \mathrm{m\,s^{-1}}$.

After Q has been moving for 5 seconds it reaches the point A.

(a) Use the equation $\mathbf{v.v} = \mathbf{u.u} + 2\mathbf{a.x}$ to show that at A the kinetic energy of Q is $37m$ J. **[5]**

(b) **(i)** Show that the power initially generated by the force is $-8m$ W. **[2]**

(ii) The power in part **(b)(i)** is negative. Explain what this means about the initial motion of Q. **[1]**

(c) **(i)** Find the time at which the power generated by the force is instantaneously zero. **[3]**

(ii) Find the minimum kinetic energy of Q in terms of m. **[2]**

a. $\underline{x} = \underline{u}t + \frac{1}{2}\underline{a}t^2$

Sub into $\underline{v}.\underline{v} = \underline{u}.\underline{u} + 2\underline{a}.\underline{x}$

$\Rightarrow \underline{v}.\underline{v} = \underline{u}.\underline{u} + 2\underline{a}.\underline{u}t + \underline{a}.\underline{a}t^2$

$= \begin{pmatrix} 2 \\ -5 \end{pmatrix} . \begin{pmatrix} 2 \\ -5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ -5 \end{pmatrix} \times 5 + \begin{pmatrix} 1 \\ 2 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \end{pmatrix} \times 5^2$

$= 4 + 25 + 10(2 - 10) + 25(1 + 4)$

$\Rightarrow \underline{v}.\underline{v} = 74$

$\therefore \underline{v} = \sqrt{74}\ ms^{-1}$

$KE = \frac{1}{2} \times m \times 74 = 37m\ J$ (as required) b.i. $P = F\underline{v}$ and $F = m\underline{a}$

$\therefore P = m\,\underline{a}\cdot\underline{v}$

$= m\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-5\end{pmatrix}$

$= m(2-10)$

$P = -8m$ W (as required)

ii. Q is slowing down (Kinetic energy is decreasing)

c.i. $\underline{v} = \underline{u} + \underline{a}.t$

$= \begin{pmatrix}2\\-5\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix}t$

$= \begin{pmatrix}2+t\\2t-5\end{pmatrix}$

at $P=0$: $m\,\underline{a}.\underline{v} = 0$

$m\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2+t\\2t-5\end{pmatrix} = 0$

$\Rightarrow 2+t+2(-5+2t)=0$

$5t = 8$

$t = 1.6$ s

ii. Minimum KE occurs when the force produces no power:

$KE = \frac{1}{2}mv^2 = \frac{1}{2}\times m\times\begin{pmatrix}2+1.6\\2\times1.6-5\end{pmatrix}\cdot\begin{pmatrix}2+1.6\\2\times1.6-5\end{pmatrix}$

$= \frac{m}{2}(12.96+3.24)$

$= 8.1m$ J

4 A right circular cone C of height 4 m and base radius 3 m has its base fixed to a horizontal plane. One end of a light elastic string of natural length 2 m and modulus of elasticity 32 N is fixed to the vertex of C. The other end of the string is attached to a particle P of mass 2.5 kg.

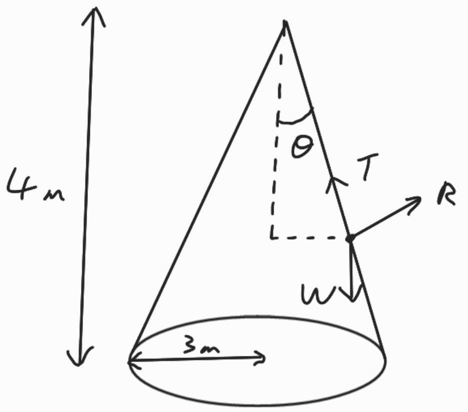
P moves in a horizontal circle with constant speed and in contact with the smooth curved surface of C. The extension of the string is 1.5 m.

(a) Find the tension in the string. **[2]**

(b) Find the speed of P. **[7]**

a. $T = \frac{\lambda x}{l} = \frac{32 \times 1.5}{2} = 24\ N$

b.



$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 36.9°$

$\therefore r = 3.5 \sin\theta = 2.1\ m$

Resolving vertically:

$T\cos\theta + R\sin\theta = mg$

$24 \times \frac{4}{5} + R \times \frac{3}{5} = 2.5 \times 9.8$

$R = \frac{53}{6}$

Resolving horizontally:

$T\sin\theta - R\cos\theta = ma$ $\quad (a = v^2/r)$

$24 \times \frac{3}{5} - R \times \frac{4}{5} = 2.5\frac{v^2}{r}$

$$\frac{72}{5} - \frac{53}{6} \times \frac{4}{5} = 2.5\frac{v^2}{2.1}$$

$$v^2 = 6.16$$

$$v = 2.48\ ms^{-1}$$

5 A particle P of mass 4.5 kg is free to move along the x-axis. In a model of the motion it is assumed that P is acted on by two forces:

- a constant force of magnitude f N in the positive x direction;
- a resistance to motion, R N, whose magnitude is proportional to the speed of P.

At time t seconds the velocity of P is $v\,\mathrm{m\,s^{-1}}$. When $t = 0$, P is at the origin O and is moving in the positive direction with speed $u\,\mathrm{m\,s^{-1}}$, and when $v = 5$, $R = 2$.

(a) Show that, according to the model, $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{10f - 4v}{45}$. **[2]**

(b) (i) By solving the differential equation in part **(a)**, show that $v = \frac{1}{2}\left(5f - (5f - 2u)\mathrm{e}^{-\frac{4}{45}t}\right)$. **[5]**

(ii) Describe briefly how, according to the model, the speed of P varies over time in each of the following cases.

- $u < 2.5f$
- $u = 2.5f$
- $u > 2.5f$

[3]

(c) In the case where $u = 2f$, find in terms of f the exact displacement of P from O when $t = 9$. **[4]**

a. $|R| \propto |v|$

$\therefore |R| = k|v|$

at $v = 5$, $R = 2$ $\quad \therefore k = \frac{2}{5} = 0.4$

$$a = \frac{\Sigma f}{m}$$

$$\frac{dv}{dt} = \frac{f - 0.4v}{4.5} \quad \left(\times \frac{10}{10}\right)$$

$$\frac{dv}{dt} = \frac{10f - 4v}{45}$$ (as required)

b. i. $$\int \frac{1}{10f-4v}\,dv = \int \frac{1}{45}\,dt$$

$$-\frac{1}{4}\ln(10f-4v) = \frac{t}{45} + c$$

$$\ln(10f-4v) = c - \frac{4t}{45}$$

$$10f - 4v = e^{\left(c - \frac{4}{45}t\right)}$$

$$10f - 4v = Ae^{-\frac{4}{45}t}$$

at $t = 0$, $v = u$:

$$10f - 4u = A$$

$$\therefore 10f - 4v = (10f - 4u)e^{-\frac{4}{45}t}$$

$$4v = 10f - (10f - 4u)e^{-\frac{4}{45}t}$$

$$v = \frac{1}{2}\left(5f - (5f - 2u)e^{-\frac{4}{45}t}\right)$$ (as required)

ii. if $u < 2.5f$, $5f - 2u > 0$

$\Rightarrow$ v increases from u and $\to 2.5f$ as $t \to \infty$

if $u = 2.5f$, $5f - 2u = 0$

$\Rightarrow$ v is constant, $= 2.5f$

if $u > 2.5f$, $5f - 2u < 0$

$\Rightarrow$ v decreases from u and $\to 2.5f$ as $t \to \infty$

c. $\dfrac{dx}{dt} = \dfrac{1}{2} f\left(5 - e^{-\frac{4}{45}t}\right)$

$\therefore x = \dfrac{1}{2} f \displaystyle\int_0^9 \left(5 - e^{-\frac{4}{45}t}\right) dt$

$x = \left[\dfrac{5}{2} ft + \dfrac{45}{8} f e^{-\frac{4}{45}t}\right]_0^9$

$x = \dfrac{5}{2} f \times 9 + \dfrac{45}{8} f e^{-\frac{4}{45}\times 9} - \left(0 + \dfrac{45}{8} f\right)$

$x = \dfrac{135}{8} f + \dfrac{45}{8} f e^{-\frac{4}{5}}$

$x = \dfrac{45}{8}\left(3 + e^{-0.8}\right) f$

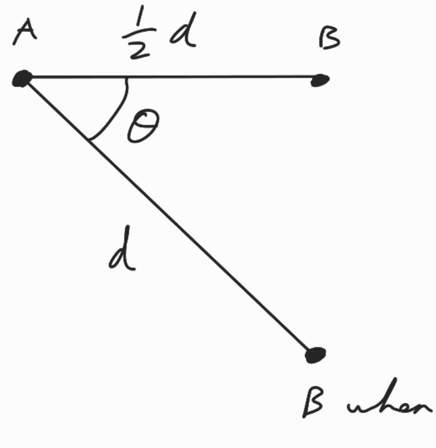
6 Two particles A and B, of masses m kg and 1 kg respectively, are connected by a light inextensible string of length d m and placed at rest on a smooth horizontal plane a distance of $\frac{1}{2}d$ m apart. B is then projected horizontally with speed $v\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to AB.

(a) Show that, at the instant that the string becomes taut, the magnitude of the instantaneous impulse in the string, I N s, is given by $I = \dfrac{\sqrt{3}mv}{2(1+m)}$. **[4]**

(b) Find, in terms of m and v, the kinetic energy of B at the instant after the string becomes taut. Give your answer as a single algebraic fraction. **[3]**

(c) In the case where m is very large, describe, with justification, the approximate motion of B after the string becomes taut. **[2]**

a. Top view:



$\cos\theta = \dfrac{1/2}{1} = \dfrac{1}{2}$

$\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$

Conservation of momentum in direction of string:

particles effectively combine

$v\sin\theta + 0 = (1+m)V$ ← they travel at a new speed, V

$$V = \frac{v \times \sqrt{3}/2}{(1+m)}$$

$$\boxed{I = mV = \frac{\sqrt{3}\,mv}{2(1+m)}}$$ (as required)

b. After string becomes taut:
Along string, speed $= V$
Perpendicular to string, speed $= v\cos\theta$

$\therefore$ overall speed $= \sqrt{V^2 + v^2\cos^2\theta}$

$$\therefore KE = \tfrac{1}{2} \times 1\left(V^2 + v^2\cos^2\theta\right)$$

$$= \frac{1}{2}\left(\left(\frac{\sqrt{3}\,v}{2(1+m)}\right)^2 + \frac{v^2}{4}\right)$$

$$= \frac{1}{2}\left(\frac{3v^2}{4(1+m)^2} + \frac{v^2}{4}\right)$$

$$= \frac{3v^2 + v^2(1+m)^2}{8(1+m)^2}$$

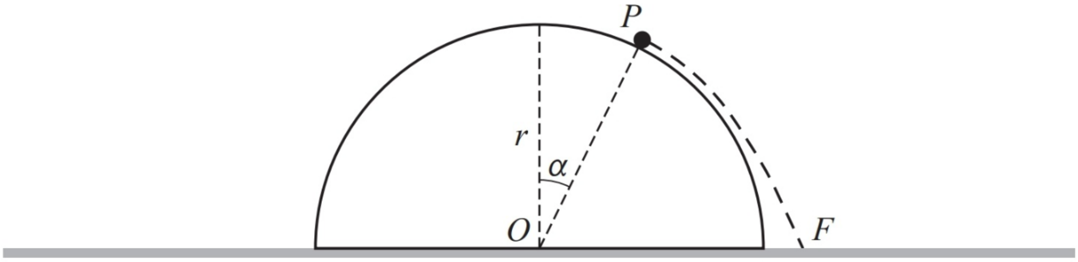
$$= \frac{3v^2 + v^2 + v^2m^2 + 2v^2m}{8(1+m)^2}$$

$$\boxed{KE = \frac{v^2(4 + 2m + m^2)}{8(1+m)^2}}$$

c. $V = \dfrac{\sqrt{3}\,v}{2(1+m)}$. $\therefore$ as $m \to \infty$, $V \to 0$

$\therefore$ B only has its velocity perpendicular to the string, and so will move in a circle around A.

7



The flat surface of a smooth solid hemisphere of radius r is fixed to a horizontal plane on a planet where the acceleration due to gravity is denoted by γ. O is the centre of the flat surface of the hemisphere.

A particle P is held at a point on the surface of the hemisphere such that the angle between OP and the upward vertical through O is α, where $\cos\alpha = \frac{3}{4}$.

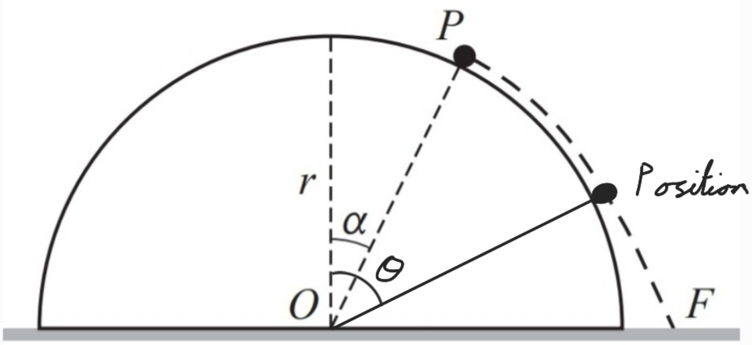
P is then released from rest. F is the point on the plane where P first hits the plane (see diagram).

(a) Find an exact expression for the distance OF. **[11]**

The acceleration due to gravity on and near the surface of the planet Earth is roughly 6γ.

(b) Explain whether OF would increase, decrease or remain unchanged if the action were repeated on the planet Earth. **[1]**

a.



Position B = P is about to loose contact with the surface

For this section: Initial energy = GPE = $m\gamma r\cos\alpha$

Final energy = GPE + KE = $m\gamma r\cos\theta + \frac{1}{2}mv^2$

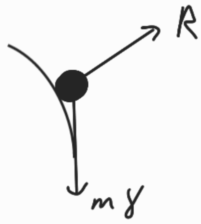
By conservation of momentum

$$\frac{1}{2}\cancel{m}v^2 + \cancel{m}\gamma r\cos\theta = \cancel{m}\gamma r\cos\alpha$$

$$\frac{v^2}{2} + \gamma r\cos\theta = \frac{3}{4}\gamma r$$

$$v^2 + 2\gamma r\cos\theta = \frac{3}{2}\gamma r \quad \text{— (1)}$$

At B:



$\therefore\ \Sigma f$ perpendicular to curve
$= m\gamma\cos\theta - R$

However P is about to loose contact, so $R = 0$

$$\Rightarrow \cancel{m}\gamma\cos\theta = \cancel{m}a$$

$$\gamma\cos\theta = a \qquad \left(\text{and } a = \frac{v^2}{r}\right)$$

$$\gamma\cos\theta = \frac{v^2}{r}$$

$$\cos\theta = \frac{v^2}{\gamma r}$$

Sub this result into (1):

$$v^2 + 2\gamma r\frac{v^2}{\gamma r} = \frac{3}{2}\gamma r$$

$$v^2 = \frac{1}{2}\gamma r \qquad \Rightarrow \cos\theta = \frac{1}{2} \quad \therefore\ \theta = 60^\circ$$

$$v = \sqrt{\frac{1}{2}\gamma r}$$

After P looses contact, suvat can be used:

$s = r\cos\theta$

$u = \sin\theta \cdot \sqrt{\tfrac{1}{2}\gamma r} = \dfrac{\sqrt{6\gamma r}}{4}$

$v = -$

$a = \gamma$

$t = t$

$s = ut + \tfrac{1}{2}at^2$

$r\cos\theta = \dfrac{\sqrt{6\gamma r}}{4}t + \dfrac{1}{2}\gamma t^2$ $(\times 2)$

$t^2\gamma + \dfrac{\sqrt{6\gamma r}}{2}t - r = 0$ $(\div \gamma)$

$t^2 + \sqrt{\dfrac{6\gamma r}{4\gamma^2}} - \dfrac{r}{\gamma} = 0$

$t^2 + \sqrt{\dfrac{3r}{2\gamma}} - \dfrac{r}{\gamma} = 0$

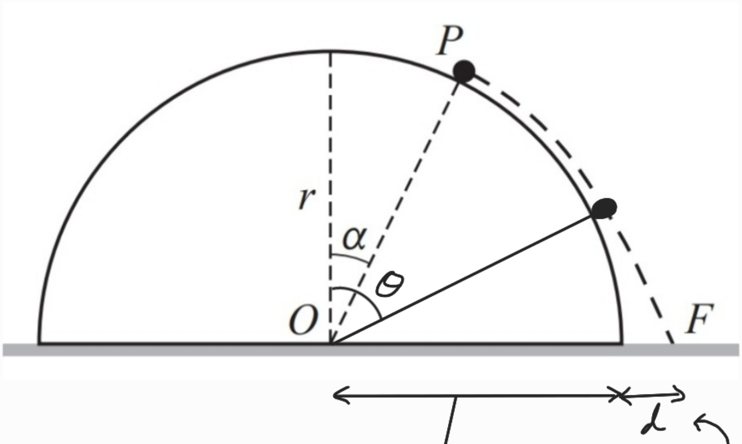
t cannot be negative

$\therefore\ t = \dfrac{-\sqrt{\frac{3r}{2\gamma}} + \sqrt{\frac{3r}{2\gamma} - 4\times 1\times\left(-\frac{r}{\gamma}\right)}}{2}$

$= \dfrac{\sqrt{\frac{r}{\gamma}}\left(-\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}+4}\right)}{2}$

$= \sqrt{\dfrac{r}{\gamma}}\left(\dfrac{-\sqrt{6}}{4} + \dfrac{\sqrt{22}}{4}\right)$

$t = \left(\dfrac{1}{4}\sqrt{\dfrac{r}{\gamma}}\left(\sqrt{22} - \sqrt{6}\right)\right)$ seconds



Now OF is simply $r\sin\theta$ + distance travelled after P looses contact

$$d = st = \sqrt{\tfrac{1}{2}\cancel{\gamma} r}\cos\theta \times \frac{1}{4}\sqrt{\frac{r}{\cancel{\gamma}}}\left(\sqrt{22} - \sqrt{6}\right)$$

$$= r\left(\frac{1}{\sqrt{2}} \times \frac{1}{2} \times \frac{1}{4} \times \left(\sqrt{22} - \sqrt{6}\right)\right)$$

$$= r\left(\frac{\sqrt{11}}{8} - \frac{\sqrt{3}}{8}\right)$$

$$\therefore Of = \frac{r\sqrt{3}}{2} + r\left(\frac{\sqrt{11}}{8} - \frac{\sqrt{3}}{8}\right)$$

$$= \frac{r}{8}\left(\sqrt{11} + 3\sqrt{3}\right)$$

b. Remain unchanged, as Of does not depend on γ.